The Malmheden Theorem and the geometry of harmonic functions

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Harmonic functions

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For all $x \in \Omega$,

$$\Delta u(x) := \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x).$$

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The Mean Value Theorem

Theorem

Let $\Omega \subseteq \mathbb{R}^n$ be open and $u \in L^1_{\text{loc}}(\Omega)$. The following conditions are equivalent:

(i). *The function u belongs to $C^2(\Omega)$ and $\Delta u = 0$ in Ω.*

(ii). *For almost every $x_0 \in \Omega$ and almost every $r > 0$ such that $B_r(x_0) \Subset \Omega$, we have that*

$$u(x_0) = \fint_{\partial B_r(x_0)} u(x)\, d\mathcal{H}^{n-1}_x.$$

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Proof of the Mean Value Theorem

If u satisfies either (ii) or (iii), it is actually $C^\infty(\Omega)$, since it coincides with its mollification u_η.

For instance, if (ii) holds,

$$\begin{aligned}
u_\eta(x) &:= \int_{B_\eta} \tau_\eta(y)\, u(x-y)\, dy \\
\text{by polar coordinates} &= \int_0^\eta \left[\int_{\partial B_\rho} \tau_\eta(\rho e_1)\, u(x-\rho\omega)\, d\mathcal{H}^{n-1}_\omega \right] d\rho \\
\text{by (ii)} &= \int_0^\eta \left[\mathcal{H}^{n-1}(\partial B_\rho) \tau_\eta(\rho e_1)\, u(x) \right] d\rho \\
\text{by polar coordinates} &= u(x) \int_{B_\eta} \tau_\eta(y)\, dy \\
&= u(x).
\end{aligned}$$

(The proof if (iii) can be done by reducing to (ii)).

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Proof (i)⇒(ii)

Take $x_0 := 0$.

$$\begin{aligned}
\partial_\rho \left(⨍_{\partial B_\rho} u(x)\, d\mathcal{H}^{n-1}_x \right) &= \partial_\rho \left(⨍_{\partial B_1} u(\rho\omega)\, d\mathcal{H}^{n-1}_\omega \right) \\
&= ⨍_{\partial B_1} \nabla u(\rho\omega) \cdot \omega \, d\mathcal{H}^{n-1}_\omega \\
&= \frac{1}{\mathcal{H}^{n-1}(\partial B_1)} \int_{\partial B_\rho} \nabla u(x) \cdot \nu(x)\, d\mathcal{H}^{n-1}_x \\
\text{by Divergence Theorem} \quad &= \frac{1}{\mathcal{H}^{n-1}(\partial B_1)} \int_{B_\rho} \Delta u(x)\, dx \\
&= 0.
\end{aligned}$$

Hence, since we know already that u is continuous (actually, smooth),

$$⨍_{\partial B_r} u(x)\, d\mathcal{H}^{n-1}_x = \lim_{\rho \searrow 0} ⨍_{\partial B_\rho} u(x)\, d\mathcal{H}^{n-1}_x = u(0).$$

Proof (ii)⇒(iii)

Use polar coordinates:

$$\fint_{B_r} u(x)\,dx = \frac{1}{|B_r|}\int_0^r \left[\int_{\partial B_\rho} u(x)\,d\mathcal{H}_x^{n-1}\right]\,d\rho$$

$$= \frac{1}{|B_r|}\int_0^r \left[\mathcal{H}^{n-1}(\partial B_\rho)\;u(0)\right]\,d\rho = \frac{\mathcal{H}^{n-1}(\partial B_1)\;u(0)}{|B_1|\;r^n}\int_0^r \rho^{n-1}\,d\rho$$

$$= \frac{\mathcal{H}^{n-1}(\partial B_1)\;u(0)\;r^n}{|B_1|\;r^n\;n} = u(0).$$

Proof (iii)⇒(i)

Use that u is smooth, a Taylor expansion and odd cancellations:

$$\begin{aligned}
0 &= \lim_{r\searrow 0} \frac{1}{r^2} \fint_{B_r} \big(u(x)-u(0)\big)\,dx \\
&= \lim_{r\searrow 0} \frac{1}{|B_r|\,r^2} \int_{B_r} \left(\nabla u(0)\cdot x + \frac{1}{2} D^2 u(0) x\cdot x + O(|x|^3)\right)\,dx \\
&= \lim_{r\searrow 0} \frac{1}{2\,|B_1|\,r^{n+2}} \int_{B_r} \left(\sum_{i=1}^n \partial_i^2 u(0)\right) x_i^2\,dx + O(r) \\
&= \text{const}\,\Delta u(0).
\end{aligned}$$

Kuran Theorem

Question: Does the mean value formula characterize the domain? If every harmonic function in Ω satisfies the mean value formula, then is it Ω necessarily a ball?

Epstein (1962), Epstein and Schiffer (1965), Goldstein and Wellington (1971), Kuran (1972).

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Let Ω be an open subset of $\mathbb{R}^n$ containing the origin and with the property that

$$u(0) = \fint_\Omega u(x)\,dx$$

for all functions u that are harmonic in Ω.

Then, Ω is a ball.

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Then, Ω is a ball.

Proof of Kuran Theorem

Up to a dilation, we suppose $B_1 \subset \Omega$ and there exists $y \in (\partial B_1) \cap (\partial \Omega)$.
Let us consider the "Poisson Kernel"

$$h(x) := \frac{|x|^2 - 1}{|x-y|^n} + 1.$$

By inspection, h is harmonic in $\mathbb{R}^n \setminus \{y\}$, $h(0) = 0$ and $h \geq 1$ in $\mathbb{R}^n \setminus B_1$.
Therefore

$$\begin{aligned} 0 = h(0) = \fint_{\Omega} h(x)\,dx &= \frac{1}{|\Omega|}\left(\int_{B_1} h(x)\,dx + \int_{\Omega\setminus B_1} h(x)\,dx\right) \\ &= \frac{1}{|\Omega|}\left(|B_1|\,h(0) + \int_{\Omega\setminus B_1} h(x)\,dx\right) = \frac{1}{|\Omega|}\int_{\Omega\setminus B_1} h(x)\,dx \\ &\geq \frac{|\Omega \setminus B_1|}{|B_1|}. \end{aligned}$$

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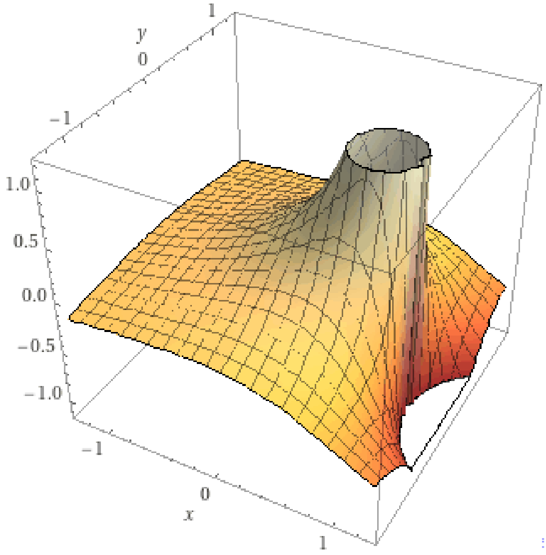
Recall: the Poisson Kernel of the ball



Recall: the Poisson Kernel of the ball

Theorem

The solution of

$$\begin{cases} \Delta u = 0 & \text{in } B_1, \\ u = f & \text{on } \partial B_1, \end{cases}$$

has the form

$$u(x) = \fint_{\partial B_1} f(y) \, \frac{1 - |x|^2}{|x - y|^n} \, d\mathcal{H}^{n-1}_y.$$

Malmheden Theorem

A “geometric argument” to construct harmonic functions in a ball with given boundary datum.

$$\begin{cases} \Delta u = 0 & \text{in } B_1, \\ u = f & \text{on } \partial B_1. \end{cases}$$

- consider a point P in the ball,
- take an arbitrary chord passing through P and calculate the value at P of the linear function that interpolates the values of f at the endpoints of the chord,
- compute the average of these values over all possible chords through P.

This procedure produces the harmonic function in the ball with datum f on the boundary.

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Malmheden Theorem

Given $P \in B_1$ and $e \in \partial B_1$, let $Q^P_+(e)$ and $Q^P_-(e)$ (or, for short, $Q_+(e)$ and $Q_-(e)$) be the intersection between the straight line of direction e passing through P and ∂B_1, that is

$$Q_+(e) = P + r_+(e)\, e$$

and $$Q_-(e) = P + r_-(e)\, e,$$

where $$r_\pm(e) := -P \cdot e \pm \sqrt{D(e)},$$

with $$D(e) := (P \cdot e)^2 - |P|^2 + 1.$$

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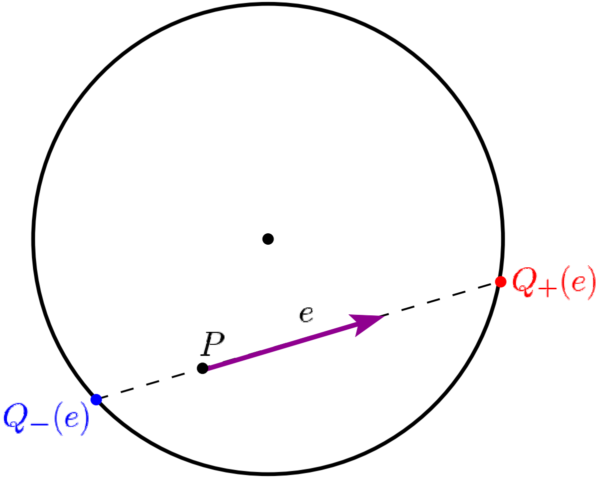

Malmheden Theorem

Let ℓ_e be the affine function on $P + e\mathbb{R}$ such that $\ell_e(Q_-(e)) = f(Q_-(e))$ and $\ell_e(Q_+(e)) = f(Q_+(e))$.

$$\ell_e(P+se) = \frac{\Big(f(Q_+(e)) - f(Q_-(e))\Big)s + r_+(e)f(Q_-(e)) - r_-(e)f(Q_+(e))}{r_+(e) - r_-(e)}.$$

Take the average over e:

$$u(P) := \fint_{\partial B_1} \ell_e(P)\, d\mathcal{H}_e^{n-1}.$$

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Malmheden Theorem

Theorem (Malmheden)

This u is the solution of the Dirichlet problem in the ball:

$$\begin{cases} \Delta u = 0 & \textit{in } B_1, \\ u = f & \textit{on } \partial B_1. \end{cases}$$

Malmheden Theorem

Note that Malmheden Theorem contains the Mean Value Theorem as a special case, by taking $P := 0$: indeed, when $P = 0$,

$$Q_+(e) = P + r_+(e)\,e,$$
$$Q_-(e) = P + r_-(e)\,e,$$
$$r_\pm(e) := -P \cdot e \pm \sqrt{D(e)}$$
$$\text{and} \qquad D(e) := (P \cdot e)^2 - |P|^2 + 1$$

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Note that Malmheden Theorem contains the Mean Value Theorem as a special case, by taking $P := 0$: indeed, when $P = 0$,

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Therefore

$$\ell_e(0) = \frac{r_+(e) f(Q_-(e)) - r_-(e) f(Q_+(e))}{r_+(e) - r_-(e)} = \frac{f(e) + f(-e)}{2}.$$

Hence,

$$u(0) = \fint_{\partial B_1} \frac{f(e) + f(-e)}{2} \, d\mathcal{H}_e^{n-1} = \fint_{\partial B_1} f(e) \, d\mathcal{H}_e^{n-1},$$

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Proof of Malmheden Theorem

The proof relies on some useful "change of variable" formulas on the sphere:

For every continuous function $g : \mathbb{R}^n \to \mathbb{R}$,

$$(1) \quad \int_{\partial B_1} g(\omega)\, d\mathcal{H}^{n-1}_\omega = \int_{\partial B_1} g\left(Q_\pm(\omega)\right) \frac{(\pm r_\pm(\omega))^n}{1 - |P|^2 - r_\pm(\omega) P \cdot \omega}\, d\mathcal{H}^{n-1}_\omega .$$

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Also,

$$\begin{aligned} u(P) &= \fint_{\partial B_1} \frac{r_+(e) f(Q_-(e)) - r_-(e) f(Q_+(e))}{r_+(e) - r_-(e)} \, d\mathcal{H}_e^{n-1} \\ &= \fint_{\partial B_1} \frac{r_+(e) f(Q_-(e))}{r_+(e) - r_-(e)} \, d\mathcal{H}_e^{n-1} - \fint_{\partial B_1} \frac{r_-(e) f(Q_+(e))}{r_+(e) - r_-(e)} \, d\mathcal{H}_e^{n-1} \\ &= 2 \fint_{\partial B_1} \frac{r_+(e) f(Q_-(e))}{r_+(e) - r_-(e)} \, d\mathcal{H}_e^{n-1}. \end{aligned} \tag{2}$$

Besides, by a direct computation,

$$\frac{2r_+(e)}{r_+(e) - r_-(e)} = \frac{1 - |P|^2}{1 - |P|^2 - (P \cdot e) r_-(e)}$$

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Thus, using (1), applied with $g(e) := \frac{f(e)\,(1-|P|^2)}{|P-e|^n}$, and (2),

$$
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u(P) &= \fint_{\partial B_1} \frac{f(Q_-(e))\,(1-|P|^2)}{1-|P|^2-(P\cdot e)r_-(e)}\, d\mathcal{H}^{n-1}_e \\
&= \fint_{\partial B_1} \frac{f(Q_-(e))\,(1-|P|^2)}{|P-Q_-(e)|^n}\, \frac{(-r_-(e))^n}{1-|P|^2-(P\cdot e)r_-(e)}\, d\mathcal{H}^{n-1}_e \\
&= \fint_{\partial B_1} g(Q_-(e))\, \frac{(-r_-(e))^n}{1-|P|^2-(P\cdot e)r_-(e)}\, d\mathcal{H}^{n-1}_e \\
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\end{aligned}
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The integrand is precisely the Poisson Kernel of the ball, hence u is the solution of the Dirichlet problem.

Proof of Malmheden Theorem

It only remains to prove (1).

This relies on the following observations: First, for every $\omega \in \partial B_1$ we have that

$$|\det DQ_\pm(\omega)| = \frac{(\pm r_\pm(\omega))^n}{1 - |P|^2 - r_\pm(\omega) P \cdot \omega}. \tag{3}$$

Also, there is a "spherical change of variable formula" for a diffeomorphism Q of $B_R \setminus B_r$ such that $Q(\partial B_\rho) = \partial B_\rho$ for each $\rho \in [r, R]$:

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To prove (4): use the classical change of variable $x := Q(y)$ to find

$$\begin{aligned}\int_{\partial B_1} g(\omega)\, d\mathcal{H}^{n-1}_\omega &= \frac{n}{R^n - r^n} \int_r^R \left[\int_{\partial B_1} \rho^{n-1} g(\omega)\, d\mathcal{H}^{n-1}_\omega \right] d\rho \\ &= \frac{n}{R^n - r^n} \int_{B_R \setminus B_r} g\left(\frac{x}{|x|} \right) dx \\ &= \frac{n}{R^n - r^n} \int_{B_R \setminus B_r} g\left(\frac{Q(y)}{|Q(y)|} \right) |\det DQ(y)|\, dy \\ &= \frac{n}{R^n - r^n} \int_r^R \left[\int_{\partial B_1} \rho^{n-1} g\left(\frac{Q(\rho\omega)}{|Q(\rho\omega)|} \right) |\det DQ(\rho\omega)|\, d\mathcal{H}^{n-1}_\omega \right] d\rho.\end{aligned}$$

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This is just careful linear algebra.
Up to a rotation, we can suppose that the points O, P and $P+\omega$ lie in the plane $\{x_3 = \cdots = x_n = 0\}$. Also, up to a further rotation in this plane, we can suppose that $\omega = e_1$. Thus,

$$Q_\pm(\omega + \varepsilon e_1) = (1+\varepsilon)\big(Q_{\pm,1}(e_1), Q_{\pm,2}(e_1), 0, \dots, 0\big).$$

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Proof of Malmheden Theorem

Also, $|\omega+\varepsilon e_j| = |e_1+\varepsilon e_j| = 1+o(\varepsilon)$, therefore

$$\begin{aligned}
& Q_\pm(\omega+\varepsilon e_j) \\
= \; & (1+o(\varepsilon))\, Q_\pm\left(\frac{\omega+\varepsilon e_j}{|\omega+\varepsilon e_j|}\right) \\
= \; & P + r_\pm\left(\frac{\omega+\varepsilon e_j}{|\omega+\varepsilon e_j|}\right)\frac{\omega+\varepsilon e_j}{|\omega+\varepsilon e_j|} + o(\varepsilon) \\
= \; & P + \left(-P\cdot\frac{\omega+\varepsilon e_j}{|\omega+\varepsilon e_j|} \pm \sqrt{\left(P\cdot\frac{\omega+\varepsilon e_j}{|\omega+\varepsilon e_j|}\right)^2 - |P|^2+1}\right)(\omega+\varepsilon e_j) + o(\varepsilon) \\
= \; & P + \left(-P\cdot(\omega+\varepsilon e_j) \pm \sqrt{(P\cdot(\omega+\varepsilon e_j))^2 - |P|^2+1}\right)(\omega+\varepsilon e_j) + o(\varepsilon) \\
= \; & P + \left(-P\cdot\omega - \varepsilon P\cdot e_j \pm \sqrt{(P\cdot\omega)^2 + 2\varepsilon(P\cdot\omega)(P\cdot e_j) - |P|^2+1}\right)(\omega+\varepsilon e_j) + o(\varepsilon) \\
= \; & P + \left(-P\cdot e_1 - \varepsilon P\cdot e_j \pm \sqrt{(P\cdot e_1)^2 - |P|^2+1} \pm \frac{\varepsilon(P\cdot e_1)(P\cdot e_j)}{\sqrt{(P\cdot e_1)^2 - |P|^2+1}}\right)(e_1+\varepsilon e_j) \\
& +o(\varepsilon).
\end{aligned}$$

Proof of Malmheden Theorem

Taking the first order in ε, we find that

$$\partial_j Q_\pm(\omega) \quad = \quad \alpha_j e_1 + r_\pm(e_1) e_j,$$

where

$$\alpha_j := \mp \frac{P \cdot e_j \, r_\pm(e_1)}{\sqrt{D(e_1)}}.$$

These observations lead to

$$DQ_\pm(\omega) = \begin{pmatrix} Q_{\pm,1}(e_1) & Q_{\pm,2}(e_1) & 0 & 0 & 0 & \dots & 0 \\ \alpha_2 & r_\pm(e_1) & 0 & 0 & 0 & \dots & 0 \\ \alpha_3 & 0 & r_\pm(e_1) & 0 & 0 & \dots & 0 \\ \alpha_4 & 0 & 0 & r_\pm(e_1) & 0 & \dots & 0 \\ \alpha_5 & 0 & 0 & 0 & r_\pm(e_1) & \dots & 0 \\ & & & \ddots & & & \\ \alpha_n & 0 & 0 & 0 & 0 & \dots & r_\pm(e_1) \end{pmatrix}$$

and therefore

$$|\det DQ_\pm(\omega)| = \left| r_\pm(e_1)^{n-2} \det \begin{pmatrix} Q_{\pm,1}(e_1) & Q_{\pm,2}(e_1) \\ \alpha_2 & r_\pm(e_1) \end{pmatrix} \right|. \tag{5}$$

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We also note that

$$\begin{aligned} Q_{\pm}(e_1) &= (P\cdot e_1, P\cdot e_2, 0,\dots,0) + \left(-P\cdot e_1 \pm \sqrt{(P\cdot e_1)^2 - |P|^2 + 1}\right) e_1 \\ &= \left(\pm\sqrt{(P\cdot e_1)^2 - |P|^2 + 1}, P\cdot e_2, 0,\dots,0\right) \\ &= \left(\pm\sqrt{D(e_1)}, P\cdot e_2, 0,\dots,0\right) \end{aligned}$$

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Proof of Malmheden Theorem

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$$\begin{aligned} Q_\pm(e_1) &= (P\cdot e_1, P\cdot e_2, 0,\dots,0) + \left(-P\cdot e_1 \pm \sqrt{(P\cdot e_1)^2 - |P|^2 + 1}\right) e_1 \\ &= \left(\pm\sqrt{(P\cdot e_1)^2 - |P|^2+1}, P\cdot e_2, 0,\dots,0\right) \\ &= \left(\pm\sqrt{D(e_1)}, P\cdot e_2, 0,\dots,0\right) \end{aligned}$$

and consequently

$$\det\begin{pmatrix} Q_{\pm,1}(e_1) & Q_{\pm,2}(e_1) \\ \alpha_2 & r_\pm(e_1)\end{pmatrix} = \frac{(r_\pm(e_1))^2}{(r_\pm(e_1))^2 + P\cdot e_1\, r_\pm(e_1)}.$$

Since

$$(r_\pm(e_1))^2 + P\cdot e_1\, r_\pm(e_1) = 1 - |P|^2 - (P\cdot e_1) r_\pm(e_1),$$

we arrive at

$$\det\begin{pmatrix} Q_{\pm,1}(e_1) & Q_{\pm,2}(e_1) \\ \alpha_2 & r_\pm(e_1)\end{pmatrix} = \frac{(r_\pm(e_1))^2}{1 - |P|^2 - (P\cdot e_1) r_\pm(e_1)}.$$

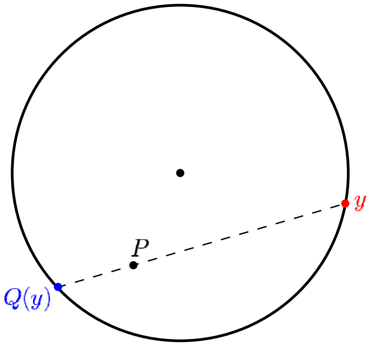
Thus, retaking (5),

$$|\det DQ_\pm(\omega)| = \left|\frac{(r_\pm(e_1))^n}{1-|P|^2-(P\cdot e_1)r_\pm(e_1)}\right|.$$

Checking the positivity of the latter term, we obtain (3).

The two-dimensional case

A particular case of Malmheden Theorem is when $n = 2$.



For this, for every $y \in \partial B_1 \subset \mathbb{R}^2$, let $Q^P(y)$, or for short $Q(y)$, be defined by

$$Q(y) := y - \frac{2(P-y)\cdot y}{|P-y|^2}(P-y).$$

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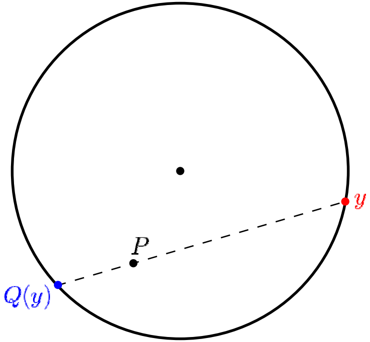

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The two-dimensional case

Theorem (Schwarz)

Let $n = 2$ *and*

$$u(P) := \fint_{\partial B_1} f(Q(\omega))\, d\mathcal{H}^1_\omega.$$

This u *is the solution of the Dirichlet problem in the ball:*

$$\begin{cases} \Delta u = 0 & \text{in } B_1, \\ u = f & \text{on } \partial B_1. \end{cases}$$

The two-dimensional case

A beautiful application of Schwarz Theorem is the determination of the temperature of a plate in which the boundary temperature is kept to 1 along an arc of circumference and to 0 along the rest of the boundary.

That is, if the temperature is 1 along an arc Σ and 0 on the rest of the circumference, can you tell me the temperature at a point P of the disk only using elementary geometry?

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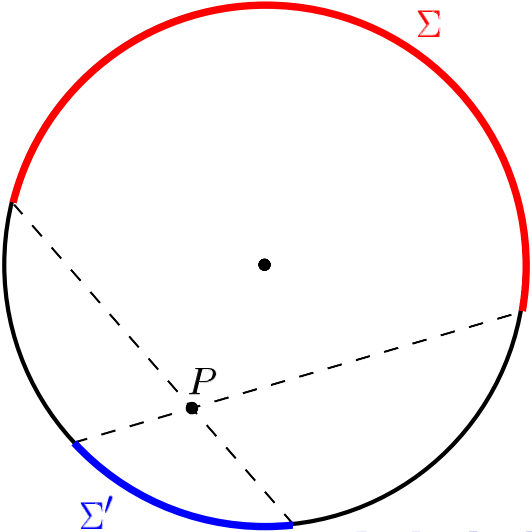

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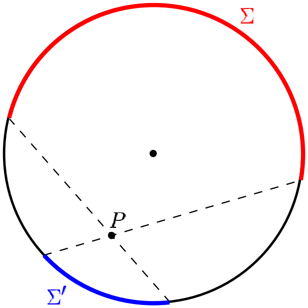

Answer: one projects the arc Σ through the focal point P obtaining a "conjugated arc" Σ'. Then, the temperature at P is exactly equal to the length of Σ' divided by 2π. Indeed this is the content of Schwarz Theorem when $f := \chi_\Sigma$.

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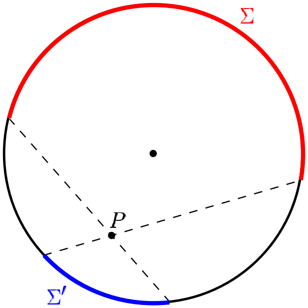

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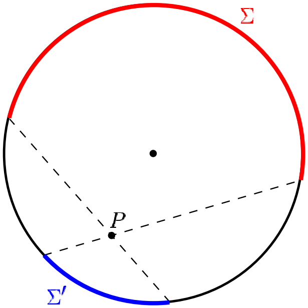

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We stress that Schwarz Theorem only holds in the plane.
Indeed, if $n \geq 3$, let $\Sigma := \partial B_1 \cap \{x_n < 0\}$ be the lower halfsphere and $f := \chi_\Sigma$.
If Schwarz Theorem held true we would have that the function $u(P)$ given by the surface area of the spherical cap obtained by projecting Σ through the point P would be harmonic.
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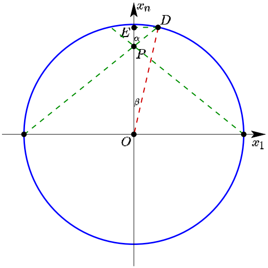
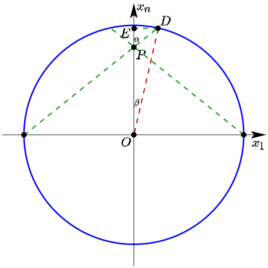


Indeed, if $\varepsilon \in (0,1)$ and $P = (0,\dots,0,1-\varepsilon)$ then (by trigonometry or "dimensional analysis") we would have that $u(P) \simeq \varepsilon^{n-1}$. Therefore

$$\partial_\nu u(0,\dots,1) = \lim_{\varepsilon\searrow 0} \frac{u(0,\dots,0,1-\varepsilon) - u(0,\dots,0,1)}{\varepsilon} = 0,$$

against Hopf Lemma.



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$$\begin{aligned}
\fint_{\partial B_1} f(Q(\omega))\, d\mathcal{H}^1_\omega &= \fint_{\partial B_1} g(\omega)\, d\mathcal{H}^1_\omega \\
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&= \fint_{\partial B_1} f\left(Q(Q_-(\omega))\right) \frac{\left(r_-(\omega)\right)^2}{1-|P|^2 - r_-(\omega) P\cdot\omega}\, d\mathcal{H}^1_\omega \\
&= \fint_{\partial B_1} f(Q_+(\omega)) \frac{\left(r_-(\omega)\right)^2}{1-|P|^2 - r_-(\omega) P\cdot\omega}\, d\mathcal{H}^1_\omega \\
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Fractional Laplacian

What is the Laplacian?

Multiplication by $|\xi|^2$ in the "frequency space".

If $\mathcal{F}$ is the Fourier transform,

$$u(x) = \int_{\mathbb{R}^n} \mathcal{F}u(\xi)\, e^{ix\cdot\xi}\, d\xi.$$

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Fractional Laplacian

Strong interest from the theoretical point of view

- harmonic analysis,
- singular integrals,
- fractional calculus,
- pseudodifferential operators...

Fractional Laplacian

...and in view of concrete applications

- finance,
- engineering,
- elasticity,
- quantum mechanics,
- fluid mechanics,
- phase transitions,
- materials sciences,
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Models:

- boundary (lower dimensional) effects,
- long-range interactions.

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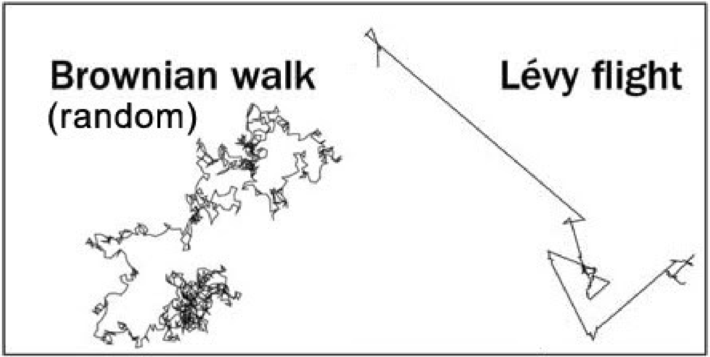
Probability:

- stochastic processes with "long jumps" (Lévy flights),
- classical processes at "discrete times" (stroboscopic lamps),
- classical processes at a "lower dimensional set" (trace theory).

Probability:



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E.g. in an integer lattice $h\mathbb{Z}^n$, jumping from hk to $h\tilde{k}$ in time step h^{2s} occurs with probability density proportional to

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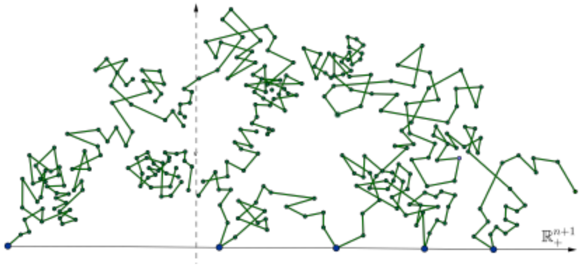
Trace/boundary stochastic processes.



An s-harmonic function

Theorem

For any $x \in \mathbb{R}$, *let* $w_s(x) := x_+^s = \max\{x, 0\}^s$. *Then*

$$(-\Delta)^s w_s(x) = \begin{cases} -c_s|x|^{-s} & \text{if } x < 0, \\ 0 & \text{if } x > 0, \end{cases} \tag{6}$$

for a suitable constant $c_s > 0$.

Proof

First, we show that

$$(7) \qquad \int_0^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\, dt + \int_1^{+\infty} \frac{(1+t)^s}{t^{1+2s}}\, dt = \frac{1}{s}.$$

Indeed, given $\varepsilon > 0$, we integrate by parts:

$$\begin{aligned}
\Xi \quad &:= \int_\varepsilon^1 \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\, dt \\
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&= \frac{1}{2s} \left[\frac{(1+\varepsilon)^s + (1-\varepsilon)^s - 2}{\varepsilon^{2s}} - 2^s + 2 \right] \\
&\qquad + \frac{1}{2} \int_\varepsilon^1 \frac{(1+t)^{s-1} - (1-t)^{s-1}}{t^{2s}}\, dt \\
&= \frac{1}{2s} [o(1) - 2^s + 2] + \frac{1}{2} \left(\int_\varepsilon^1 (1+t)^{s-1} t^{-2s}\, dt - \int_\varepsilon^1 (1-t)^{s-1} t^{-2s}\, dt \right).
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Proof

Moreover, by changing variable $\tilde{t} := t/(1-t)$,

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$$\begin{aligned} \Xi &= \frac{1}{2s}\left[o(1) - 2^s + 2\right] \\ &\quad + \frac{1}{2}\left[\int_{\varepsilon}^{1} (1+t)^{s-1} t^{-2s}\, dt - \int_{\varepsilon/(1-\varepsilon)}^{+\infty} (1+t)^{s-1} t^{-2s}\, dt\right] \\ &= \frac{1}{2s}\left[o(1) - 2^s + 2\right] \\ &\quad + \frac{1}{2}\left[\int_{\varepsilon}^{\varepsilon/(1-\varepsilon)} (1+t)^{s-1} t^{-2s}\, dt - \int_{1}^{+\infty} (1+t)^{s-1} t^{-2s}\, dt\right]. \end{aligned}$$

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Also, since

$$\begin{aligned}\int_{\varepsilon}^{\varepsilon/(1-\varepsilon)} (1+t)^{s-1} t^{-2s}\,dt &\le \int_{\varepsilon}^{\varepsilon/(1-\varepsilon)} (1+\varepsilon)^{s-1} \varepsilon^{-2s}\,dt \\ &= \varepsilon^{2-2s}(1-\varepsilon)^{-1}(1+\varepsilon)^{s-1},\end{aligned}$$

we have

$$\lim_{\varepsilon \searrow 0} \int_{\varepsilon}^{\varepsilon/(1-\varepsilon)} (1+t)^{s-1} t^{-2s}\,dt = 0.$$

Therefore

$$\Xi = \frac{-2^s+2}{2s} - \frac{1}{2}\int_1^{+\infty} (1+t)^{s-1} t^{-2s}\,dt.$$

Now, integrating by parts,

$$\begin{aligned}\frac{1}{2}\int_1^{+\infty} (1+t)^{s-1} t^{-2s}\,dt &= \frac{1}{2s}\int_1^{+\infty} t^{-2s}\frac{d}{dt}(1+t)^s\,dt \\ &= -\frac{2^s}{2s} + \int_1^{+\infty} t^{-1-2s}(1+t)^s\,dt.\end{aligned}$$

Proof

Also, since

$$\int_{\varepsilon}^{\varepsilon/(1-\varepsilon)} (1+t)^{s-1} t^{-2s}\, dt \leq \int_{\varepsilon}^{\varepsilon/(1-\varepsilon)} (1+\varepsilon)^{s-1} \varepsilon^{-2s}\, dt = \varepsilon^{2-2s} (1-\varepsilon)^{-1} (1+\varepsilon)^{s-1},$$

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Proof

Therefore,

$$\int_0^1 \frac{(1+t)^s+(1-t)^s-2}{t^{1+2s}}\,dt = \frac{-2^s+2}{2s}+\frac{2^s}{2s}-\int_1^{+\infty} t^{-1-2s}(1+t)^s\,dt,$$

proving (7).

Proof

Now, we claim that

$$(-\Delta)^s w_s(1) = 0. \tag{8}$$

the function $t \mapsto (1+t)^s + (1-t)^s - 2$ is even, therefore

$$\int_{-1}^{1} \frac{(1+t)^s + (1-t)^s - 2}{|t|^{1+2s}}\, dt = 2\int_{0}^{1} \frac{(1+t)^s + (1-t)^s - 2}{t^{1+2s}}\, dt.$$

Moreover, by changing variable $\tilde{t} := -t$,

$$\int_{-\infty}^{-1} \frac{(1-t)^s - 2}{|t|^{1+2s}}\, dt = \int_{1}^{+\infty} \frac{(1+\tilde{t})^s - 2}{\tilde{t}^{1+2s}}\, d\tilde{t}.$$

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Proof

Therefore,

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{w_s(1+t)+w_s(1-t)-2w_s(1)}{|t|^{1+2s}}\,dt \\
&= \int_{-\infty}^{-1} \frac{(1-t)^s-2}{|t|^{1+2s}}\,dt + \int_{-1}^{1} \frac{(1+t)^s+(1-t)^s-2}{|t|^{1+2s}}\,dt \\
&\qquad + \int_{1}^{+\infty} \frac{(1+t)^s-2}{|t|^{1+2s}}\,dt \\
&= 2\int_0^1 \frac{(1+t)^s+(1-t)^s-2}{t^{1+2s}}\,dt + 2\int_1^{+\infty} \frac{(1+t)^s-2}{t^{1+2s}}\,dt \\
&= 2\left[\Xi + \int_1^{+\infty} \frac{(1+t)^s}{t^{1+2s}}\,dt - 2\int_1^{+\infty} \frac{dt}{t^{1+2s}}\right] \\
&= 2\left[\frac{1}{s} - 2\int_1^{+\infty} \frac{dt}{t^{1+2s}}\right].
\end{aligned}$$

Proof

Since

$$\int_1^{+\infty} \frac{dt}{t^{1+2s}} = \frac{1}{2s},$$

we obtain that

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Moreover,

$$w_s(-1+t) + w_s(-1-t) - 2w_s(-1) = (-1+t)_+^s + (-1-t)_+^s \geq 0$$

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Proof

Now, we let $\sigma \in \{+1, -1\}$ denote the sign of a fixed $x \in \mathbb{R} \setminus \{0\}$. We claim that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{w_s(\sigma(1+t)) + w_s(\sigma(1-t)) - 2w_s(\sigma)}{|t|^{1+2s}}\,dt \\
&= \int_{-\infty}^{+\infty} \frac{w_s(\sigma+t) + w_s(\sigma-t) - 2w_s(\sigma)}{|t|^{1+2s}}\,dt.
\end{aligned} \tag{10}$$

Indeed, the formula above is obvious when $x > 0$ (i.e. $\sigma = 1$), so we suppose $x < 0$ (i.e. $\sigma = -1$) and we change variable $\tau := -t$:

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&\int_{-\infty}^{+\infty} \frac{w_s(\sigma(1+t)) + w_s(\sigma(1-t)) - 2w_s(\sigma)}{|t|^{1+2s}}\,dt \\
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Now we use a scaling argument: for any $r \in \mathbb{R}$,

$$w_s(|x|r) = (|x|r)_+^s = |x|^s r_+^s = |x|^s w_s(r).$$

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We say that u is s-harmonic in Ω if $(-\Delta)^s u = 0$ in Ω.

An arbitrary function can be approximated arbitrarily well in a given ball by functions whose fractional Laplacian vanishes in such ball, in sharp contrast with the rigidity of classical harmonic functions.

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All functions are s-harmonic:

S. Dipierro, O. Savin, E. Valdinoci (2017)

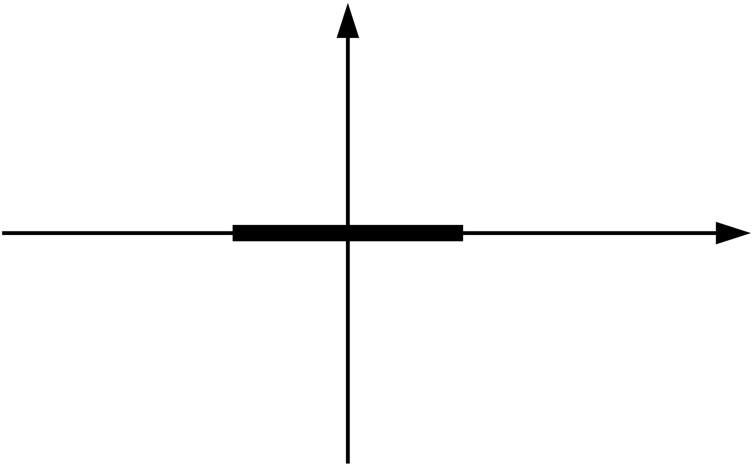
Fix $k \in \mathbb{N}$. Then, given any function $u \in C^k(B_1)$ and any $\varepsilon > 0$, there exist $R_\varepsilon > 1$ and $u_\varepsilon \in H^s(\mathbb{R}^n) \cap C^s(\mathbb{R}^n)$ such that

$$(-\Delta)^s u_\varepsilon = 0 \quad \text{in } B_1,$$

$$u_\varepsilon = 0 \quad \text{in } \mathbb{R}^n \setminus B_{R_\varepsilon}$$

and $$\|u - u_\varepsilon\|_{C^k(B_1)} \leq \epsilon.$$

All functions are s-harmonic:



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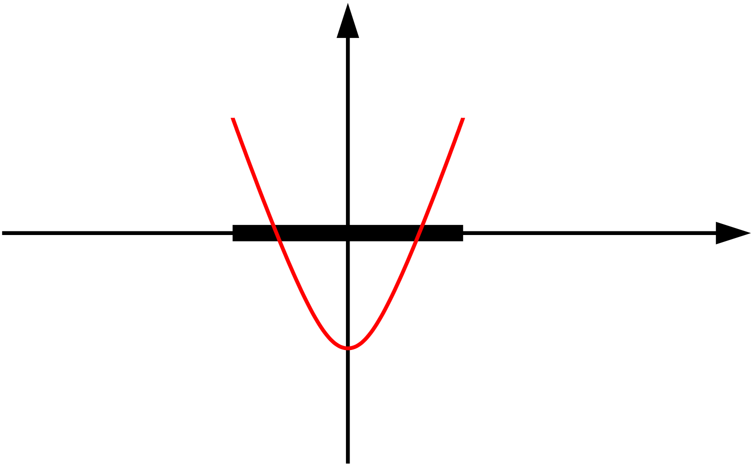

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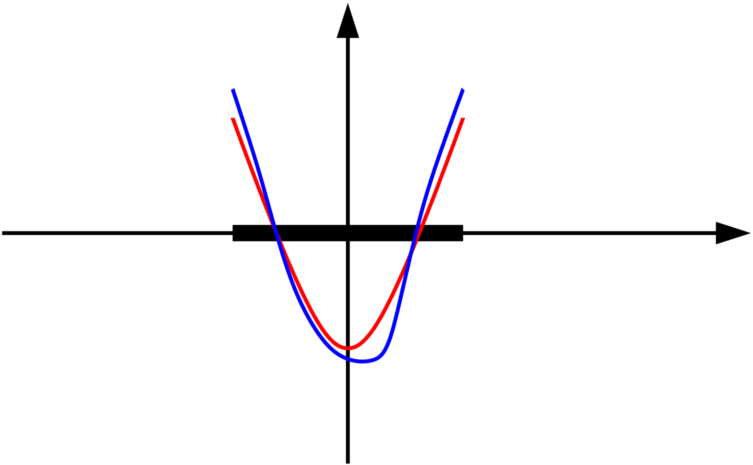

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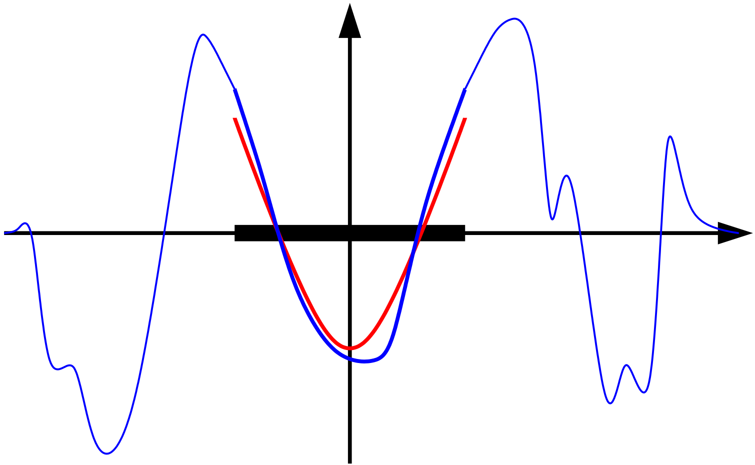

Sketch of the proof:

By Stone-Weierstrass Theorem, it suffices to prove the result for polynomials. Hence, from now on, we suppose that

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Core Lemma: spanning the derivative of a function.

There exists a function v such that

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Sketch of the proof:

Proof of the Core Lemma in $1D$:

Let $\mathcal{Z}$ be the set of functions v such that $(-\Delta)^s v = 0$ in $(-r,r)$. For any $v \in \mathcal{Z}$, let

$$\mathcal{D}^\beta v(0) := \big(v(0), v'(0), \dots, v^\beta(0)\big) \in \mathbb{R}^{\beta+1}.$$

Let

$$V := \{\mathcal{D}^\beta v(0) \text{ s.t. } v \in \mathcal{Z}\}.$$

Notice that $V \subseteq \mathbb{R}^{\beta+1}$ is a vector space.

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By contradiction, we suppose that there exists $c = (c_j) \in \mathbb{R}^{\beta+1} \setminus \{0\}$ such that

$$\sum_{j=0}^{\beta} c_j v^j(0) = 0.$$

Choose $v(x) := (x + \eta)^s_+$, then

$$0 = \sum_{j=0}^{\beta} c_j v^j(0) = \sum_{j=0}^{\beta} c_j s(s-1)\cdots(s-j+1)\,\eta^{s-j}$$

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Sketch of the proof:

Mean Value Theorem
Converse Mean Value Theorem
Malmheden Theorem
Schwarz Theorem
Fractional Malmheden Theorem
Fractional Schwarz Theorem
Superposition Theorem
Fractional Harnack Inequality

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Set $t := 1/\eta$:

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Use the Identity Principle of Polynomials and therefore, for every $j \in \{0, \ldots, \beta\}$,

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Define

$$u_\varepsilon(x) := \frac{v(\varepsilon x)}{\varepsilon^{|\beta|}}.$$

Since

$$v(x) = \frac{x^\beta}{\beta!} + O(x^\gamma),$$

with $|\gamma| > |\beta|$, we have that

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Fractional Mean Value Theorem

Theorem

A function $u : \mathbb{R}^n \to \mathbb{R}$ is s-harmonic in B_R if and only if, for each $r \in (0, R)$,

$$u(0) = c(n,s) \int_{\mathbb{R}^n \setminus B_r} \frac{r^{2s}\, u(y)}{(|y|^2 - r^2)^s |y|^n}\, dy,$$

where

$$c(n,s) := \left(\int_{\mathbb{R}^n \setminus B_1} \frac{dy}{(|y|^2 - 1)^s |y|^n} \right)^{-1}.$$

We define

$$d\mu_r(y) := \frac{c(n,s)\, r^{2s}\, dy}{(|y|^2 - r^2)^s |y|^n}$$

and we can interpret μ_r as a probability measure on $\mathbb{R}^n \setminus B_r$. Then, the Fractional Mean Value Theorem can be written, for short,

$$u(0) = \int_{\mathbb{R}^n \setminus B_r} u(y)\, d\mu_r(y).$$

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Fractional Kuran Theorem

The Fractional Mean Value Theorem has an “inverse”, in the spirit of Kuran Theorem:

C. Bucur, S. Dipierro, E. Valdinoci (2020)

Let $\Omega \subset \mathbb{R}^n$ *be a bounded open set, containing the origin, and* $r := \mathrm{dist}(0, \partial\Omega)$.
Suppose that

$$u(0) = \frac{1}{\mu_r(\mathbb{R}^n \setminus \Omega)} \int_{\mathbb{R}^n \setminus \Omega} u(y)\, d\mu_r(y) \tag{11}$$

for all s*-harmonic functions* u *in* Ω.
Then, $\Omega = B_r$.

In short: if Ω satisfies a fractional mean value property with respect to a suitable measure, then Ω is necessarily a ball.

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First proof of Fractional Kuran Theorem

By contradiction, assume that $\Omega \setminus B_r \neq \varnothing$ and pick $p \in \Omega \setminus B_r$. Since Ω is open, there exists $\rho > 0$ such that $B_\rho(p) \subset \Omega$ and thus

$$\varnothing \neq B_\rho(p) \setminus \overline{B_r} \subset \Omega \setminus B_r.$$

Therefore,

$$\mu_r(\Omega \setminus B_r) > 0.$$

Moreover, if u is s-harmonic in Ω with $u(0) = 0$,

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First proof of Fractional Kuran Theorem

Now we use that "all functions are locally s-harmonic up to an ϵ error", with $\epsilon := \frac{r^2}{4}$, our "favorite function $f(x) := |x|^2$, and a reference domain B_R, with $R := \max_{y \in \overline{\Omega}} |y|$.
So, we find $f_{r,R}$ such that

$$(-\Delta)^s f_{r,R} = 0 \quad \text{in } B_R,$$

$$\text{and} \qquad \|f_{r,R} - f\|_{L^\infty(B_R)} \leq \epsilon = \frac{r^2}{4}.$$

Then, we define

$$u^\star(x) := -f_{r,R}(x) + f_{r,R}(0).$$

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First proof of Fractional Kuran Theorem

For all $x \in B_R$,

$$\begin{aligned} u^\star(x) &= -f_{r,R}(x) + f(x) + f_{r,R}(0) - f(0) - f(x) + f(0) \\ &\leq -f(x) + f(0) + |f(x) - f_{r,R}(x)| + |f_{r,R}(0) - f(0)| \\ &\leq -|x|^2 + \frac{r^2}{2}. \end{aligned}$$

Hence, for all $x \in B_R \setminus B_r$,

$$-u^\star(x) \geq |x|^2 - \frac{r^2}{2} \geq \frac{r^2}{2}.$$

Since $\Omega \subset B_R$, it follows that

$$\int_{\Omega \setminus B_r} -u^\star(y)\, d\mu_r(y) \geq \frac{r^2}{2}\, \mu_r(\Omega \setminus B_r) > 0.$$

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First proof of Fractional Kuran Theorem

Since $u^\star(0) = 0$ and $(-\Delta)^s u^\star(x) = (-\Delta)^s f_{r,R}(x) = 0$ for all $x \in B_R$,

$$0 = -\int_{\Omega \setminus B_r} u^\star(y)\, d\mu_r(y) > 0,$$

contradiction!

Second proof of Fractional Kuran Theorem

A structurally different proof of the Fractional Kuran Theorem is based on potential theory.

The idea is to suppose for simplicity (after a suitable approximation) that Ω has $C^{1,1}$ boundary, and use the fractional Poisson Kernel: we know that the fractional Poisson Kernel of $B_R(x_0)$ is

$$P_{B_R(x_0)}(x,y) = \frac{c(n,s)\,(R^2 - |x-x_0|^2)^s}{(|y-x_0|^2 - R^2)^s |x-y|^n}$$

and if u is s-harmonic in Ω then

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$$u(x) = \int_{\mathbb{R}^n \setminus \Omega} u(y)\, P_\Omega(x,y)\, dy.$$

Second proof of Fractional Kuran Theorem

Hence, under assumption (11), for any $u \in C_0^\infty(\mathbb{R}^n \setminus \Omega)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus\Omega} u(y) \left(\frac{P_{B_r}(0,y)}{\mu_r(\mathbb{R}^n \setminus \Omega)} - P_\Omega(0,y) \right) dy \\
= &\ \frac{1}{\mu_r(\mathbb{R}^n \setminus \Omega)} \int_{\mathbb{R}^n\setminus\Omega} u(y)\, P_{B_r}(0,y)\, dy - u(0) \\
= &\ 0.
\end{aligned}$$

Hence, by the arbitrariness of u,

$$\frac{P_{B_r}(0,y)}{\mu_r(\mathbb{R}^n \setminus \Omega)} = P_\Omega(0,y) \tag{12}$$

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This is an identity about fractional Poisson Kernels and we need to show that it forces Ω to be the ball B_r.

Suppose not. The idea is then to choose a point $p^* \in \partial\Omega$, and $p^* \notin \partial B_r$ and take the limit for $y \in \mathbb{R}^n \setminus \Omega$ to p^*: the right-hand side of (12) will tend to infinity, whereas the left-hand side gives a finite value.

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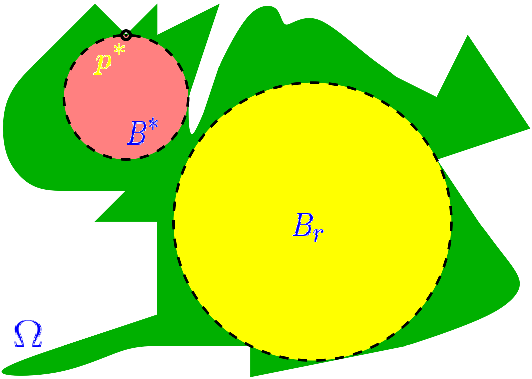
Second proof of Fractional Kuran Theorem

The technical details go as follows: take a ball $B^* \subset \Omega \setminus \overline{B_r}$ with $(\partial B^*) \cap ((\partial\Omega) \setminus \overline{B_r}) \neq \varnothing$ and pick a point $p^* \in (\partial B^*) \cap ((\partial\Omega) \setminus \overline{B_r})$. We also take a sequence $p_j \in \mathbb{R}^n \setminus \Omega$ such that $p_j \to p^*$ as $j \to +\infty$.

Second proof of Fractional Kuran Theorem



Second proof of Fractional Kuran Theorem

Let $\varpi := B_r \cup B^*$. We consider, as a test for our contradiction, a harmonic function in ϖ formally corresponding to a Dirac mass at p_j.

On the one hand, this function will reproduce the Poisson Kernel $P_\varpi(\cdot, p_j)$; on the other hand, the corresponding average would converge to a finite value, thus providing the desired contradiction.

The details of the technical argument go as follows. We take $\varphi \in C_0^\infty(B_1, [0,1])$, with φ even and $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$.

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Second proof of Fractional Kuran Theorem

Let

$$\varphi_{k,p}(x) := k^n \varphi\left(k(x-p)\right)$$

and $u_{k,p}$ such that

$$\begin{cases} (-\Delta)^s u_{k,p} = 0 & \text{in } \varpi, \\ u_{k,p} = \varphi_{k,p} & \text{in } \mathbb{R}^n \setminus \varpi. \end{cases}$$

Given j, we always suppose that k is large, possibly in dependence of j, such that

$$\overline{B_{1/k}(p_j)} \subset \mathbb{R}^n \setminus \overline{\Omega}.$$

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Given $\delta > 0$, we take a smooth bounded open set $\Omega^{(\delta)}$ that contains Ω and such that all points of $\Omega^{(\delta)}$ have distance less than δ from Ω.

We take δ sufficiently small (possibly in dependence of k and j), such that

$$\overline{B_{1/k}(p_j)} \subset \mathbb{R}^n \setminus \overline{\Omega^{(\delta)}}.$$

We take $u_{k,p_j,\delta}$ to be the fractional harmonic function coinciding with φ_{k,p_j} outside $\Omega^{(\delta)}$.

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We claim that

$$u_{k,p_j,\delta} \geq u_{k,p_j}.$$

Indeed, $u_{k,p_j,\delta} \geq 0$, by Maximum Principle. Hence, since $u_{k,p_j} = \varphi_{k,p_j} = 0$ in $(\mathbb{R}^n \setminus \varpi) \cap (\mathbb{R}^n \setminus B_{1/k}(p_j))$, it follows that the claim holds true at least in $(\mathbb{R}^n \setminus \varpi) \cap (\mathbb{R}^n \setminus B_{1/k}(p_j)) \supseteq (\mathbb{R}^n \setminus \varpi) \cap \Omega^{(\delta)}$. Since, by construction, it holds true in $\mathbb{R}^n \setminus \Omega^{(\delta)}$, it holds true in $\mathbb{R}^n \setminus \varpi$.

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Second proof of Fractional Kuran Theorem

Now we claim that

$$\lim_{\delta\to 0}\int_{\mathbb{R}^n\setminus\Omega} u_{k,p_j,\delta}(y)\,d\mu_r(y)=\int_{\mathbb{R}^n\setminus\Omega}\varphi_{k,p_j}(y)\,d\mu_r(y).$$

To this end, we observe that the image of φ_{k,p_j} is $[0,k]$, and therefore also the image of $u_{k,p_j,\delta}$ is $[0,k]$, by Maximum Principle. Then, since

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus\Omega} u_{k,p_j,\delta}(y)\,d\mu_r(y)\\ =\;&\int_{\mathbb{R}^n\setminus\Omega^{(\delta)}} u_{k,p_j,\delta}(y)\,d\mu_r(y)+\int_{\Omega^{(\delta)}\setminus\Omega} u_{k,p_j,\delta}(y)\,d\mu_r(y)\\ =\;&\int_{\mathbb{R}^n\setminus\Omega^{(\delta)}} \varphi_{k,p_j}(y)\,d\mu_r(y)+\int_{\Omega^{(\delta)}\setminus\Omega} u_{k,p_j,\delta}(y)\,d\mu_r(y),\end{aligned}$$

one obtains the claim by taking the limit.

Therefore,

$$\begin{aligned} u_{k,p_j}(0) &\le \lim_{\delta\to 0} u_{k,p_j,\delta}(0) \\ &= \lim_{\delta\to 0} \frac{1}{\mu_r(\mathbb{R}^n \setminus \Omega)} \int_{\mathbb{R}^n\setminus\Omega} u_{k,p_j,\delta}(y)\, d\mu_r(y) \\ &= \frac{1}{\mu_r(\mathbb{R}^n \setminus \Omega)} \int_{\mathbb{R}^n\setminus\Omega} \varphi_{k,p_j}(y)\, d\mu_r(y). \end{aligned}$$

Second proof of Fractional Kuran Theorem

Hence, since $0 \in B_r \subset \varpi$,

$$\begin{aligned} P_\varpi(0, p_j) &= \lim_{k\to+\infty} u_{k,p_j}(0) \\ &\le \lim_{k\to+\infty} \frac{1}{\mu_r(\mathbb{R}^n \setminus \Omega)} \int_{\mathbb{R}^n\setminus\Omega} \varphi_{k,p_j}(y)\, d\mu_r(y) \\ &= \frac{c(n,s)\, r^{2s}}{\mu_r(\mathbb{R}^n \setminus \Omega)\, (|p_j|^2 - r^2)^s |p_j|^n}. \end{aligned}$$

Now, we use the geometry of the fractional Poisson Kernel, which gives a suitable $c := c(n, s, \varpi) > 0$ such that

$$P_\varpi(0, p_j) \ge \frac{c\,\big(\operatorname{dist}(0, \partial\varpi)\big)^s}{\big(\operatorname{dist}(p_j, \partial\varpi)\big)^s \big(1 + \operatorname{dist}(p_j, \partial\varpi)\big)^s |p_j|^n}.$$

Hence, since $0 \in B_r \subset \varpi$,

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Second proof of Fractional Kuran Theorem

Hence, since $p_j \to p^* \in \partial B^* \subseteq \partial \varpi$,

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and therefore

$$\begin{aligned} +\infty &= \lim_{j\to+\infty} \frac{c(n,s)\, r^{2s}}{\mu_r(\mathbb{R}^n \setminus \Omega)\,(|p_j|^2 - r^2)^s |p_j|^n} \\ &= \frac{c(n,s)\, r^{2s}}{\mu_r(\mathbb{R}^n \setminus \Omega)\,(|p^*|^2 - r^2)^s |p^*|^n} \\ &< +\infty, \end{aligned}$$

since $p^* \in \mathbb{R}^n \setminus \overline{B_r}$, contradiction.

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Fractional Malmheden Theorem

A "geometric argument" to construct s-harmonic functions in a ball with given "boundary" datum.

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_1, \\ u = f & \text{in } \mathbb{R}^n \setminus B_1. \end{cases}$$

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To establish a fractional counterpart of Malmheden Theorem one needs the following structural modifications:

- the classical spherical averages are replaced by suitable weighted spherical averages,
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Setting:

$$f_\rho(x) := f(\rho x),$$

$\mathcal{L}_f^{a,b}(x) :=$ the affine function such that

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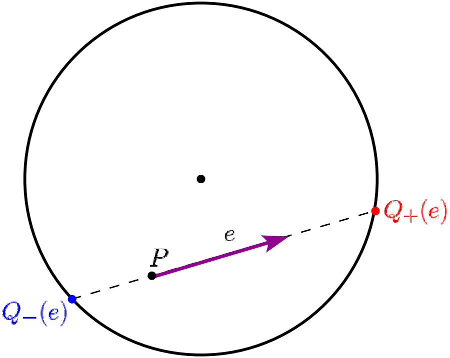
Fractional Malmheden Theorem

Recall:

$$Q_\pm(e) = P + r_\pm(e)\, e, \qquad r_\pm(e) := -P \cdot e \pm \sqrt{D(e)},$$
$$D(e) := (P \cdot e)^2 - |P|^2 + 1.$$



Fractional Malmheden Theorem

Let us also consider the following kernel for the ball B_ρ.
Given $s \in (0,1)$, let

$$\mathcal{E}(x,\rho) := c_{n,s} \frac{\rho\,(1-|x|^2)^s}{(\rho^2-1)^s\,(\rho^2-|x|^2)},$$

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$$c_{n,s} := \frac{\Gamma(n/2)\,\sin(\pi s)}{\pi^{(n+2)/2}}.$$

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Fractional Malmheden Theorem

Theorem (S. Dipierro, G. Giacomin, E. Valdinoci)

Let

$$u(P) := \int_1^{+\infty} \left[\int_{\partial B_1} \mathcal{L}_{f,e,\rho}(x)\, \mathcal{E}(x,\rho)\, d\mathcal{H}_e^{n-1} \right] d\rho.$$

Then, u is the solution of the fractional Dirichlet problem in the ball:

$$\begin{cases} (-\Delta)^s u = 0 & \textit{in } B_1, \\ u = f & \textit{in } \mathbb{R}^n \setminus B_1. \end{cases}$$

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When $P = 0$, we have that $D(e) = 1$, $r_\pm(e) = \pm 1$ and $Q_\pm(e) = \pm e$. Hence,

$$\mathcal{L}_{f,e,\rho}(0) = \mathcal{L}_{f_\rho}^{Q_-^0(e),Q_+^0(e)}(0) = \mathcal{L}_{f_\rho}^{-e,e}(0) = \frac{f(\rho e)}{2} + \frac{f(-\rho e)}{2}.$$

Also,

$$\mathcal{E}(0,\rho) = c_{n,s}\frac{\rho}{(\rho^2-1)^s\rho^2} = \frac{c_{n,s}}{(\rho^2-1)^s\rho}.$$

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Fractional Malmheden Theorem

Thus, the Fractional Malmheden Theorem reduces to

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\begin{aligned}
u(0) &= \int_1^{+\infty} \left[\int_{\partial B_1} \mathcal{L}_{f,e,\rho}(0)\, \mathcal{E}(0,\rho)\, d\mathcal{H}_e^{n-1} \right] d\rho \\
&= \int_1^{+\infty} \left[\int_{\partial B_1} \frac{c_{n,s}}{(\rho^2-1)^s \rho} \left(\frac{f(\rho e)}{2} + \frac{f(-\rho e)}{2} \right) d\mathcal{H}_e^{n-1} \right] d\rho \\
&= \int_1^{+\infty} \left[\int_{\partial B_\rho} \frac{c_{n,s}}{(\rho^2-1)^s \rho^n} \left(\frac{f(\omega)}{2} + \frac{f(-\omega)}{2} \right) d\mathcal{H}_\omega^{n-1} \right] d\rho \\
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which is the Fractional Mean Value Theorem

Fractional Malmheden Theorem

Thus, the Fractional Malmheden Theorem reduces to

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u(0) &= \int_1^{+\infty} \left[\int_{\partial B_1} \mathcal{L}_{f,e,\rho}(0)\, \mathcal{E}(0,\rho)\, d\mathcal{H}_e^{n-1} \right] d\rho \\
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Fractional Malmheden Theorem

Also, when $s \nearrow 1$, the Fractional Malmheden Theorem recovers the classical one.

Proof of Fractional Malmheden Theorem

Once we "guess" the right formula, the proof relies on a series of identities due to the geometry of the projections.

Sketch:

- Start with the representation of the s-harmonic function in B_1 via the fractional Poisson Kernel.
- This produces an integral outside B_1, which can be written in polar coordinates.
- After some scaling, one can recognize the function $\mathcal{E}$ as a weight for the integral.
- Name whatever remains g and apply to it the spherical change of variable.
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Fractional Schwarz Theorem

Theorem (S. Dipierro, G. Giacomin, E. Valdinoci)

Let $n = 2$ *and*

$$u(x) := \int_1^{+\infty} \left[\int_{\partial B_1} f_\rho\big(Q^{x/\rho}(e)\big)\, \mathcal{E}(x,\rho)\, d\mathcal{H}^1_e \right] d\rho.$$

Then, u *is the solution of the fractional Dirichlet problem in the ball:*

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_1, \\ u = f & \text{in } \mathbb{R}^n \setminus B_1. \end{cases}$$

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Fractional Schwarz Theorem

When $s \nearrow 1$, the Fractional Schwarz Theorem recovers the classical one.

Also, let Σ be an arc in the circle $\partial B_1 \subset \mathbb{R}^2$ and $\Sigma_\star$ be the cone generated by Σ, i.e.

$$\Sigma_\star := \left\{ x \in \mathbb{R}^n \setminus \{0\} \text{ s.t. } \frac{x}{|x|} \in \Sigma \right\}.$$

Consider the solution of the fractional Dirichlet problem in the ball with conical exterior datum:

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_1, \\ u = \chi_{\Sigma_\star} & \text{in } \mathbb{R}^n \setminus B_1. \end{cases}$$

Do we have a geometric way to represent such a solution?

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Mean Value Theorem
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Malmheden Theorem
Schwarz Theorem
Fractional Malmheden Theorem
Fractional Schwarz Theorem
Superposition Theorem
Fractional Harnack Inequality

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Let $|\Sigma'_{x/\rho}|$ be its length.
Then, the Fractional Schwarz Theorem yields that

$$u(x) = \int_1^{+\infty} |\Sigma'_{x/\rho}| \, \mathcal{E}(x,\rho)\, d\rho.$$

That is, u is a superposition of scaled arc lengths, weighted by a the kernel $\mathcal{E}$.
In particular,

$$u(0) = |\Sigma| \int_1^{+\infty} \mathcal{E}(0,\rho)\, d\rho = \frac{|\Sigma|}{2\pi}.$$

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Proof of Fractional Schwarz Theorem

Use the Fractional Malmheden Theorem with $n = 2$ and check the algebra.

Superposition Theorem

As a byproduct of the Malmheden Theorem and the Fractional Malmheden Theorem, we see that an s-harmonic function is the superposition of classical harmonic functions.

Superposition Theorem

Theorem (S. Dipierro, G. Giacomin, E. Valdinoci)

For each $\rho > 1$ be u_{f_ρ} be the unique solution to the classical Dirichlet problem in the ball

$$\begin{cases} \Delta u_{f_\rho} = 0 & \text{in } B_1, \\ u_{f_\rho} = f_\rho & \text{on } \partial B_1. \end{cases}$$

Then, the solution of the fractional Dirichlet problem in the ball

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B_1, \\ u = f & \text{in } \mathbb{R}^n \setminus B_1. \end{cases}$$

can be written as

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Fractional Harnack Inequality

As a consequence of the Superposition Theorem, we have a new proof of the Fractional Harnack Inequality (see e.g. M. Kaßmann's thesis), with optimal constants:

Theorem

If u is s-harmonic in B_1, then, for each $r \in (0,1)$ and $x \in B_r$,

$$\frac{(1-r^2)^s}{(1+r)^n} u(0) \le u(x) \le \frac{(1-r^2)^s}{(1-r)^n} u(0).$$

The constants above are optimal, and for $s \nearrow 1$ they converge to the optimal constants of the classical Harnack inequality in B_r for harmonic functions in B_1.

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Proof of Fractional Harnack Inequality

Applying the Harnack inequality for classical harmonic functions to u_{f_ρ}, we have that

$$u_{f_\rho}(0) \le \frac{(1+|x|/\rho)^{n-1}}{1-|x|/\rho} u_{f\rho}\left(\frac{x}{\rho}\right).$$

From this, the Malmheden Theorem and the Fractional Malmheden Theorem we obtain that

$$\begin{aligned} u(0) &\le c_{n,s}|\partial B_1| \int_1^\infty \frac{1}{\rho(\rho^2-1)^s} \frac{(1+|x|/\rho)^{n-1}}{1-|x|/\rho} u_{f\rho}\left(\frac{x}{\rho}\right) d\rho \\ &= |\partial B_1| \int_1^\infty \mathcal{E}(x,\rho) \frac{(\rho^2-|x|^2)}{\rho^2(1-|x|^2)^s} \frac{(\rho+|x|)^{n-1}}{\rho^{n-2}(\rho-|x|)} u_{f\rho}\left(\frac{x}{\rho}\right) d\rho \\ &= |\partial B_1| \int_1^\infty \mathcal{E}(x,\rho) \frac{(\rho+|x|)^n}{\rho^n(1-|x|^2)^s} u_{f\rho}\left(\frac{x}{\rho}\right) d\rho \\ &= |\partial B_1| \int_1^\infty \mathcal{E}(x,\rho)\, g(\rho,t)\, u_{f_\rho}\left(\frac{x}{\rho}\right) d\rho, \end{aligned} \tag{13}$$

where $t := |x|$ and

$$g(\rho,t) := \frac{(\rho+t)^n}{\rho^n(1-t^2)^s}.$$

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Proof of Fractional Harnack Inequality

Since $g(\rho, t)$ is decreasing in ρ and increasing in t, we have that

$$\frac{(1+r)^n}{(1-r^2)^s} = \sup_{(\rho,t)\in[1,\infty)\times[0,r]} g(\rho, t).$$

Therefore, it follows from (13) that

$$u(0) \leq |\partial B_1| \frac{(1+r)^n}{(1-r^2)^s} \int_1^\infty \mathcal{E}(x,\rho)\, u_{f_\rho}\left(\frac{x}{\rho}\right) d\rho = \frac{(1+r)^n}{(1-r^2)^s} u(x),$$

which establishes one side of the Fractional Harnack Inequality.

Proof of Fractional Harnack Inequality

To prove the other side, use again the Harnack inequality for harmonic functions:

$$u_{f_\rho}\left(\frac{x}{\rho}\right) \leq \frac{1+|x|/\rho}{(1-|x|/\rho)^{n-1}} u_{f_\rho}(0).$$

Therefore,

$$\begin{aligned} u(0) &= c_{n,s}|\partial B_1| \int_1^\infty \frac{u_{f_\rho}(0)}{\rho(\rho^2-1)^s}\, d\rho \\ &\geq c_{n,s}|\partial B_1| \int_1^\infty \frac{1}{\rho(\rho^2-1)^s} \frac{(1-|x|/\rho)^{n-1}}{1+|x|/\rho} u_{f_\rho}\left(\frac{x}{\rho}\right) d\rho \\ &= c_{n,s}|\partial B_1| \int_1^\infty \frac{(\rho-|x|)^{n-1}}{\rho^{n-1}(\rho^2-1)^s(\rho+|x|)} u_{f_\rho}\left(\frac{x}{\rho}\right) d\rho \\ &= |\partial B_1| \int_1^\infty \mathcal{E}(x,\rho) \frac{(\rho-|x|)^n}{\rho^n(1-|x|^2)^s} u_{f_\rho}\left(\frac{x}{\rho}\right) d\rho. \end{aligned}$$

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\end{aligned}$$

Proof of Fractional Harnack Inequality

Hence, if

$$g_1(\rho, t) := \frac{(\rho - t)^n}{\rho^n (1 - t^2)^s},$$

we have that

$$u(0) \geq |\partial B_1| \int_1^{\infty} \mathcal{E}(x, \rho) g_1(\rho, t)\, u_{f_\rho}\left(\frac{x}{\rho}\right) d\rho. \tag{14}$$

Since g_1 is increasing in ρ, for all $(\rho, t) \in [1, \infty) \times [0, r]$,

$$g_1(\rho, t) \geq g_1(1, t) = \frac{(1-t)^n}{(1-t^2)^s} = \frac{(1-t)^{n-s}}{(1+t)^s} =: g_2(t).$$

Since also g_2 is decreasing, for all $(\rho, t) \in [1, \infty) \times [0, r]$,

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we have that

$$u(0) \geq |\partial B_1| \int_1^\infty \mathcal{E}(x,\rho) g_1(\rho,t)\, u_{f_\rho}\left(\frac{x}{\rho}\right)\, d\rho. \tag{14}$$

Since g_1 is increasing in ρ, for all $(\rho,t) \in [1,\infty)\times[0,r]$,

$$g_1(\rho,t) \geq g_1(1,t) = \frac{(1-t)^n}{(1-t^2)^s} = \frac{(1-t)^{n-s}}{(1+t)^s} =: g_2(t).$$

Since also g_2 is decreasing, for all $(\rho,t) \in [1,\infty)\times[0,r]$,

$$g_1(\rho,t) \geq g_2(r) = \frac{(1-r)^{n-s}}{(1+r)^s} = \frac{(1-r)^n}{(1-r^2)^s}.$$

Proof of Fractional Harnack Inequality

Plugging this information into (14),

$$u(0) \geq |\partial B_1| \frac{(1-r)^n}{(1-r^2)^s} \int_1^{\infty} \mathcal{E}(x,\rho)\, u_{f_\rho}\left(\frac{x}{\rho}\right)\, d\rho \geq \frac{(1-r)^n}{(1-r^2)^s} u(x),$$

which is the other side of the Fractional Harnack Inequality.

Optimality of the constants in the Fractional Harnack Inequality

Let $\varepsilon \in (0,1)$. Then,

$$u_\varepsilon(x) := c_{n,s} \int_{B_\epsilon((\epsilon+1)e)} \frac{(1-|x|^2)^s}{|y-x|^n}\, dy$$

is s-harmonic in B_1.

Let $x = -re$ for $r \in (0,1)$: we have that

$$\frac{u_\varepsilon(0)}{u_\varepsilon(-re)} = \frac{\displaystyle\int_{B_\epsilon((\epsilon+1)e)} \frac{dy}{|y|^n}}{\displaystyle\int_{B_\epsilon((\epsilon+1)e)} \frac{(1-r^2)^s}{|y+re|^n}\, dy},$$

whence

$$\lim_{\epsilon \to 0} \frac{u_\varepsilon(0)}{u_\varepsilon(-re)} = \frac{(1+r)^n}{(1-r^2)^s},$$

showing the optimality of the constants.

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showing the optimality of the constants.

Thanks a lot for your attention!